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Flow Optimization in Dynamic Networks on Time Scales

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Abstract. Network flow optimization has a wide range of real world applications such as in transportation, in electric, in civil engineering, in industrial engineering and in communication networks and so on. In the minimum cost network flow model, the goal is to find the values of the decisions variables that minimize the total cost of flows over the given network. In this work, a new formulation of flow optimization in dynamic networks using time scales approach has been presented. The continuous network model and the quantum case model are also obtained as special cases. The formulation has been given for both dynamic models and time scales models. Moreover, the new approach provides the exact optimal solution for this type of optimization problems. Furthermore, a new version of some duality theorems for time scales flow optimization in dynamic networks has been introduced.

1. Introduction

Discrete network flow models form a large class of optimization problem and have used to model problems in operations research, physics, mathematics, and some related fields of engineering. Communication, transportation, and manufacturing systems are typical real life applications of this type of optimization problem. The minimum cost flow problem and the shortest path problem are two well-known network flow models. These problems have been studied by many researchers and many algorithms for solving this type of optimization problem have considered. For more details discussion of theory, algorithms and applications of network flow problems we refer to [3, 21, 23, 24, 28, 37{39].

On the other hand, some real world applications require flow value on an arc changes over time. This type of optimization is called dynamic network flow problem. Ford and Fulkerson [22] were first considered the dynamic network flows. The aim of this model is to obtain the maximum flow in a given network from a source node t to a sink node s with a specific time horizon T. This problem has various applications in many real world problems. Anderson et al. [11] studied the problem of maximizing the flow in dynamic network with time varying arc capacities and they also presented computational approach as well as duality theory for the dynamic network flows. Hoppe [27] introduced some efficient algorithms to find the optimal solution for dynamic network flows models.

A general dynamic network flows with arc time-delays has been presented by Pullan [36]. Orda and Rom [33] considered some new algorithms to solve the dynamic min cost problem. Minimum-cost dynamic flows problem has been presented by Klinz and Woeginger [30]. Hooks and Patterson [26] presented new formulation for dynamic networks. Recently, Nasrabadi [32] studied dynamic minimum cost flow problem and also established some of the duality theorems as well as some computational approaches. For more details about the dynamic network flows problems we refer to [3,31,34,35]. On the other hand, time scales theory has been used to formulate and solve many dynamical models. See

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for e.g. [1,2,6,9,10,13,20,25,29]. Also, linear and fractional linear programming models have been formulated using time scales. See e.g. [4,5,7,8] In this work we present time scales formulation for dynamic network flow model.

2. Time scales calculus

In this section, some basic concepts of dynamic equations on time scales are presented. The material of this section is taken from [18, 19].

Definition 1 "A time scale T is an arbitrary nonempty closed subset of the real numbers".

Definition 2 " The forward and backward jump operators $\rho, \sigma : T \to T$ are defined by $\sigma(t) = inf\, \{s \in T : s > t\}$ and $\rho(t) = sup\, \{s \in T : s < t\}$

Definition 3 The graininess function $\mu : T \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$"

Definition 4 " A function $f : [a, b]_\Gamma \to R$ is $\triangle$-differentiable at $t \in T^\kappa$ provided there exists an $\alpha \in R$ such that for each $\varepsilon > 0$, there exists a neighborhood B of t such that

$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \quad$ for all $\quad s \in B$.

α denoted by $f^\triangle(t)$ "

Definition 5 " A function $f : T \to R$ is called regulated if its right-sided limits exist (finite) at all right-dense points in T and its left-sided limits exist (finite) at all left-dense points T"

Definition 6 "A function: $T \to R$ is called rd-continuous if it is continuous at each right-dense point $t \in T$ and left-hand limits exist at each left-dense point $t \in T$"

Theorem 7 "Every rd-continuous function has an anti-derivative. In particular, if $t_0 \in T$, then F defined by

$F(t) = \int_{t_0}^{t} f\,(s) \triangle s,$

for $t \in T$ is an anti-derivative of f"

Theorem 8. "If $f : T \to R$ rd-continuous function and $t \in T^\kappa$, then

$$\int_t^{\sigma(t)} f\,(\tau) \triangle \tau = \mu(t)f(t)$$

Theorem 9. "*If $f : T \to R$ is rd-continuous and $a, b \in T$, then*

1- If $T = R$, then

$$\int_a^b f\,(t) \triangle t = \int_a^b f\,(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.

2- If $[a, b] = \{t \in T : a \le t \le b\}$ consists of only isolated points, then

$$\int_a^b f\,(t) \triangle t = \{\sum \mu\,(t)f(t)\ if\ a < b\ 0\ if\ a = b\ - \sum \mu\,(t)f(t)\ if\ a > b.$$

3- If $T = hZ = \{hk : k \in Z\}$ where $h > 0$, then

$$\int_a^b f\,(t) \triangle t = \{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} h\,f(kh)\ if\ a < b, 0\ if\ a = b, \quad -\sum_{k=\frac{a}{h}-1}^{\frac{b}{h}} \quad hf(kh)\ if\ a > b.$$

4- If $T = Z$, then

$$\int_a^b f\,(t) \triangle t = \{\sum_{t=0}^{b-1} f\,(t)\ if\ a < b, \quad 0\ if\ a = b\,, \quad -\sum_{t=b}^{a-1} f\,(t)\ if\ a > b.$$

3. Dynamic network flow model

In this section, we present dynamic network flows models. As in [31],"we denote flow over time in the network G with time horizon T is a bounded function measurable functions on $[0, T]$" " An initial storage of $u_i(0)$, $g_{i,j}(t)$ the amount of flow per time unit into arc (i,j) at time t, $c_i(t)$ represents the supply or demand rate at node i at time t, $f_i(t)$ denotes the maximum storage allowed, and $b_i(t)$ is the cost per time unit for storing one unit of flow at node i at time t" " $a(t)$, $b(t)$, $c(t)$, $g(t)$, and $f(t)$ are bounded measurable functions on $[0,T]$" The primal continuous minimum cost flow model $(PCMC\Gamma M)$ is formulated in [31] as "

$$\{ Min \ \ \varphi(x) = \int_0^T a'(t)x(t)dt + \int_0^T b'(t)u(t)dt$$

$$s.t. \int_0^t \sum_{(j:(i,j)\in A} x_{i,j}(s)ds - \int_0^t \sum_{(j:(j,i)\in A} x_{j,i}(s-\lambda i,j)ds + u_i(t) = c_i(t)\ , i \in N, t \in [0,T] \ - \int_0^t \sum_{(j:(i,j)\in A} x_{i,j}(s)ds - \int_0^t \sum_{(j:(j,i)\in A} x_{j,i}(s-\lambda i,j)ds + u_i(t) \le f_i(t) - c_i(t)\ , i \in N, t \in [0,T] \ and \ x,y \in E_n, x(t) \ge g(t), x(t), y(t) \ge 0 \ t \in [0,T]",$$

$$(PCMC\Gamma M)$$

The dual continuous minimum cost flow model $(DCMC\Gamma M)$ is formulated in [31] as"

$$\{ Max \ \ G(y,v,w) = \int_0^T c'(t)y(t)dt + \int_0^T \big(f(t)-c(t)\big)' v(t)dt + \int_0^T g'(t)w(t)dt$$

$$subject\ to \int_t^T y_i(s) - y_j\big(s+\lambda_{i,j}\big)ds - \int_t^T v_i(s) - v_j\big(s+\lambda_{i,j}\big)ds + , w_{i,j} \le a_{i,j}, (i,j) \in A, t \in [0,T] \ and \ y \in E_m, y(t), v\big(t, w(t)\big) \ge 0, t \in [0,T]"$$

$$(DCMC\Gamma M)$$

4. Network flow on time scales

We considerE_k, represents the space of all rd-continuous functions from J into R^k .The primal time scales minimum cost flow model $(PTSMC\Gamma M)$ is formulated as

$$\{ Min \ \ \varphi(x) = \int_0^{\sigma(T)} a'(t)x(t)\delta t + \int_0^{\sigma(T)} b'(t)u(t)\delta t$$

$$s.t. \int_0^t \sum_{(j:(i,j)\in A} x_{i,j}(s)\delta s - \int_0^t \sum_{(j:(j,i)\in A} x_{j,i}(s-\lambda i,j)\delta s + y_i(t) = c_i(t)\ , i \in N, t \in J \ - \int_0^t \sum_{(j:(i,j)\in A} x_{i,j}(s)\delta s - \int_0^t \sum_{(j:(j,i)\in A} x_{j,i}(s-\lambda i,j)\delta s + u_i(t) \le f_i(t) - c_i(t)\ , i \in N, t \in J \ and \ x,y \in E_n, x(t) \ge g(t), x(t), y(t) \ge 0 \ t \in J,$$

$$(PTSMCFM)$$

The dual time scales minimum cost flow model $(DTSMC\Gamma M)$ is formulated as

$$\{ Max \ \ G(y,v,W) = \int_0^{\sigma(T)} c'(t)y(t)\delta t + \int_0^{\sigma(T)} \big(f(t)-c(t)\big)' v(t)\delta t + \int_0^{\sigma(T)} g'(t)w(t)\delta t$$

$$subject\ to \int_{\sigma(t)}^{\sigma(T)} y_i(s) - y_j\big(s+\lambda_{i,j}\big)\delta s - \int_{\sigma(t)}^{\sigma(T)} v_i(s) - v_j\big(s+\lambda_{i,j}\big)\delta s + , w_{i,j} \le a_{i,j}, (i,j) \in A, t \in J \ and \ y \in E_m, y(t), v\big(t, w(t)\big) \ge 0, t \in J.$$

$$(DTSMCFM)$$

Remark1. If $T = R$, then the continuous time dynamic network flows model is obtained as described in Section 3.

Remark2. If $T = q^{N_0}$, then the quantum calculus dynamic network flows model is obtained see [11].

5. Duality Theory on Time Scales

In this section, we present a new version of the duality theorems on time scales. Moreover, the duality theorems of continuous and quantum calculus versions are obtained as special cases.

Theorem.1. If (x, u) and (y, v, w) are any two feasible solutions of the primal time scales network ftow model $(PTSMC\Gamma M)$ and the dual time scales network flow model $(DTSMCFM)$, respectively, then the following inequality hold $\varphi(x, u) \geq G(y, v, w)$.

Proof. Let (x, u) and (y, v, w) be arbitrary feasible solutions of the primal time scales network flow model $(PTSMC\Gamma M)$ and the dual time scales network flow model $(DTSMCTM)$, respectively, using Definition 2.10, the objective function of primal time scales network flow model $(PTSMCFM)$ becomes

$$\{ \ Min \ \emptyset(X, u) = \sum_{k=0}^{N} \mu t_k a'(t_k) x(t_k) + \sum_{k=0}^{N} \mu t_k b'(t_k) u(t_k) \ .$$

Using [12, Theorem 4.3 on page 146] , there exists a feasible solution (α, β, γ) of

$$\{ \ Max \ M(\alpha, \beta, \gamma) = \sum_{k=0}^{N} c'(t_k)(y(t_k) + \sum_{k=0}^{N} \big(f(t_k) - c(t_k)\big)' \beta(t_k) + \sum_{k=0}^{N} g'(t_k)\gamma(t_k) \tag{5.1}$$

and we have

$$\varphi(x, u) \geq M(\alpha, \beta, \gamma) \ . \tag{5.2}$$

Now we put

$$y(t_k) = \frac{\alpha(t_k)}{\mu t_k} \ \text{for } k = 0,1, \ldots, N. \tag{5.3}$$

$$v(t_k) = \frac{\beta(t_k)}{\mu t_k} \ \text{for } k = 0,1, \ldots, N. \tag{5.4}$$

$$w(t_k) = \frac{\gamma(t_k)}{\mu t_k} \text{for } \ k = 0,1, \ldots, N. \tag{5.5}$$

Using (5.3), (5.4), (5.5) in (5.2), we get

$$\emptyset(x, u) \geq M(\alpha, \beta, \gamma) = \sum_{k=0}^{N} c'(t_k) \left(y(t_k) + \sum_{k=0}^{N} \big(f(t_k)\big) - c(t_k) \right)' \beta(t_k) + \sum_{k=0}^{N} g'(t_k)\gamma(t_k)$$

$$= \sum_{k=0}^{N} c'(t_k)\mu(t_k) y(t_k) + \sum_{k=0}^{N} \big(f(t_k) - c(t_k)\big)' \mu(t_k)$$

$v(t_k) + \sum_{k=0}^{N} g'(t_k)\mu(t_k)w(t_k) = G(y, v, w)$.

This completes the proof.

Theorem 2. *If (x^*, u^*) is a feasible solution of the primal problem $(PTSMC\Gamma M)$ and (y^*, v^*, w^*) is a feasible solution of the dual problem $(DTSMCFM)$ with*
$\varphi(x^*, u^*) = G(y^*, v^*, w^*)$, then (x^*, u^*) is an optimal for the primal network model and (y^*, v^*, w^*) is an optimal for the dual network model.

Proof. By Theorem 5.1, any feasible solution (x, u) of the primal problem $(PTSMC\Gamma M)$,we have $\int_0^{\sigma(T)} a'(t)x(t)\delta t + \int_0^{\sigma(T)} b'(t)u(t)\delta t \geq \int_0^{\sigma(T)} c'(t)y^*(t)\delta t + \int_0^{\sigma(T)} \big(f(t) - c(t)\big)' v^*(t)\delta t + \int_0^{\sigma(T)} g'(t)w^*(t)\delta t$. This implies $\varphi(x, u) \geq G(y^*, v^*, w^*)$, but $G(y^*, v^*, w^*) = \varphi(x^*, u^*)$ is given. Therefore, $\varphi(x^*, u^*) \leq \varphi(x, u)$ for any feasible solution (x, u) of the primal problem $(PTSMC\Gamma M)$.

Hence, it follows from the definition of optimality that (x^*, u^*) is an optimal solution of the primal problem $(PTSMC\Gamma M)$. Similarly, Theorem 5.1, for any feasible solution (y, v, w) of the dual problem $(DTSMC\Gamma M)$, we have $\int_0^{\sigma(T)} a'(t)x^*(t)\delta t + \int_0^{\sigma(T)} b'(t)u^*(t)\delta t \geq \int_0^{\sigma(T)} c'(t)y(t)\delta t + \int_0^{\sigma(T)}(f(t) - c(t))'v(t)\delta t + \int_0^{\sigma(T)} g'(t)w(t)\delta t$. This implies $(x^*, u^*) \geq G(y, v, w)$, but $G(y^*, v^*, w^*) = \varphi(x^*, u^*)$ is given. Therefore, $G(y^*, v^*, w^*) \geq G(y, v, w)$ for any feasible solution (y, v, w) of the dual problem $(DTSMC\Gamma M)$. Thus, it follows from the definition of optimality that (y^*, v^*, w^*) is an optimal solution of the dual problem $(DTSMCFM)$. This completes the proof.

Theorem.3. *If the primal time scales network flow model* $(PTSMC\Gamma M)$ *has an optimal solution* (x^*, u^*) , *then the dual time scales network flow model* $(DTSMC\Gamma M)$ *has an optimal solution* (y^*, v^*, w^*) *such that* $\varphi(x^*, u^*) = G(y^*, v^*, w^*)$.

Proof. Let (x, u) be an arbitrary feasible solution of the primal quantum network flow model $(PTSMCFM)$, using Definition 2.10, the objective function of primal quantum network flow model $(PTSMC\Gamma M)$ becomes

$$\{ \ Min \ \varphi(x,u) = \sum_{k=0}^{N} \mu(t_k)a'(t_k)x(t_k) + \sum_{k=0}^{N} \mu(t_k)b'(t_k)u(t_k) \, .$$

Using [12, Theorem4.4 on page 148], there exists an optimal solution $(\alpha^*, \beta^*, \gamma^*)$ of

$$\{ \ Max \ M(\alpha, \beta, \gamma) = \textstyle\sum_{K=0}^{N} c'(t_k)\alpha(t_k) + \sum_{K=0}^{N}\big(f(t_k) - c(t_k)\big)'\beta(t_k) + \sum_{K=0}^{N} g'(t_k)\gamma(t_k)$$
(5.6)

and we have

$$\varphi(x^*, u^*) = M(\alpha^*, \beta^*, \gamma^*) \quad (5.7)$$

Now we put

$$y^*(t_k) = \frac{\alpha^*(t_k)}{\mu(t_k)} \text{ for } k = 0,1, \ldots, N. \qquad (5.8)$$

$$v^*(t_k) = \frac{\beta^*(t_k)}{\mu(t_k)} \text{ for } k = 0,1 \ldots\ldots N \qquad (5.9)$$

$$w^*(t_k) = \frac{\gamma^*(t_k)}{\mu(t_k)} \text{ for } k = 0,1, \ldots, N. \qquad (5.10)$$

Using (5.8), (5.9), (5. 10) in (5.7), we get

$$\varphi(x^*, u^*) = M(\alpha^*, \beta^*, \gamma^*) = \sum_{K=0}^{N} c'(t_k)\alpha^*(t_k) + \sum_{K=0}^{N}\big(f(t_k) - c(t_k)\big)'\beta^*(t_k) + \sum_{K=0}^{N} g'(t_k)\gamma^*(t_k)$$

$$= \sum_{K=0}^{N} c'(t_k)\mu(t_k)y^*(t_k) + \sum_{K=0}^{N}\big(f(t_k) - c(t_k)\big)'\mu(t_k)$$

$v^*(t_k) + \sum_{K=0}^{N} g'(t_k)\mu(t_k)w^*(q^k) = G(y^*, v^*, w^*)$.

Using Theorem 5.2, we obtain that (y^*, v^*, w^*) is an optimal solution for the dual network flow model. This completes the proof.

6. Conclusion

In this paper, we introduce a new version of the dynamic network model using time scales analogue. The new formulation is presented for both the primal and the dual network flow models. In addition, we introduce a new version of some duality theorems using arbitrary time scales set. Moreover, this approach is efficient and simple to implement. Furthermore, the new approach does not require any theoretical convergence results, because it gives an exact optimal solution for network flow models which reduced the large computation effort. The discrete network flow model and quantum calculus network flow model can be obtained as special cases of this formulation.

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